A Local Nonlinear System Model of the Turbulent Boundary Layer

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ABSTRACT
The interaction between the near-wall and logarithmic layer in a high Reynolds number zero pressure gradient turbulent boundary layer is examined in terms of polyspectral measurements. The near wall velocity fluctuation skewness is decomposed in frequency space via the real part of the autobispectrum. Similarly, the cross-bicoherence is used to examine the nonlinear phase coupling between the near wall region and logarithmic layer. These measurements clearly show the importance of quadratically nonlinear mechanisms in characterizing the near wall-outer layer interaction. To this end, a Volterra nonlinear system model containing both linear and quadratically nonlinear system transfer functions is proposed to model the near-wall outer layer interaction. It is demonstrated that the transfer functions may be determined via measured polyspectral quantities and the system output due to linear, quadratic and linear-quadratic coupling mechanisms quantified.

INTRODUCTION
It is now widely accepted that large-scale vortical structures are an important and universal feature of the outer region of wall bounded turbulent flows (e.g. Hutchins and Marusic (2007), Adrian (2007), Monty et al (2007), Marusic et al (2010), Smits, McKeon and Marusic (2011) and others). It has also been demonstrated that these outer layer structures impose their imprint on the near-wall region of the turbulent boundary layer in the form of the amplitude and phase modulation of near-wall velocity and wall shear stress fluctuations (e.g. Hutchins and Marusic (2007), Mathis et al (2009), Ganipathisubramani et al (2009)). In addition, Mathis et al (2011) demonstrate that the skewness of velocity fluctuations in the near-wall region is directly linked to amplitude modulation.

The nature of the interaction between outer and inner regions of wall bounded flows is of both fundamental and practical interest since the near-wall region is responsible for turbulence production. Schoppa and Hussain (2002) described a streak transient growth (STG) mechanism for the self-sustaining mechanism of near-wall turbulence generation. They suggested that STG was the dominant streamwise vortex generation mechanism from otherwise normal mode stable low-speed streaks. In related work, Schoppa and Hussain (1998) proposed a large-scale strategy for skin friction drag reduction which was demonstrated in channel flow DNS. By imposing a streamwise-independent, spanwise velocity component along the channel wall by means of either a pair of counter-rotating streamwise vortices, or opposed wall jets significant drag reduction was achieved. The flow control served to prevent the lift-up of low-speed streaks, thereby limiting their flanking wall-normal vorticity component which in their formulation, is a critical parameter for onset of STG. More recently, their channel flow drag reduction work was revisited by Canton et al (2016), where comparable levels of sustained drag reduction were achieved by a volumetric forcing approach. Motivated by this work a novel, nonintrusive, flush surface-mounted pulsed-DC plasma actuator was recently designed at the University of Notre Dame to be the first to actively intervene in the STG mechanism by producing a near-wall spanwise flow component that prevents the lift-up of low-speed streaks. Experiments show the ability of the actuator to very signficantly decrease or increase drag depending on the magnitude of the imposed spanwise velocity (Thomas et al (2016)).

Aside from the practical aspects pertaining to drag reduction, the pulsed-DC actuator also provides an experimental tool by which the nature of the outer region-inner layer interaction can be investigated under both reduced and enhanced drag conditions and compared to the natural flow. Furthermore, in order to characterize the dynamic interaction between the near-wall and outer regions of the TBL a second-order Volterra nonlinear system model is applied. This model involves the determination of both linear and nonlinear system transfer functions that characterize the interaction. A particular focus of this paper is to describe and motivate the application of the nonlinear system model to the turbulent boundary layer.

EXPERIMENTAL FACILITY
The turbulent boundary layer measurements were conducted in a large in-draft wind tunnel facility located at the University of Notre Dame with a test section cross sectional area of 1.5 m x 1.5 m, a working test section length of 15.25m, and a maximum speed of approximately 13.5m/s. The inlet is 3.05 m x 3.05 m and corresponds to a contraction ratio of 4:1. A screen box is placed behind the inlet consisting of 4 stainless steel turbulence reduction screens (mesh spacing = 1 mm and wire diameter = 0.1 mm) with...
0.229 m streamwise spacing between individual screens and 0.204 m thick honeycomb (0.013 m diameter) placed directly upstream of the screens. The resulting free stream turbulence intensity measured 9.14 m downstream of the inlet was $\sqrt{u'^{2}/U_{\infty}} \leq 0.5\%$. The wind tunnel is constructed of 1.9 cm plywood and also has a plexiglass side-wall for full optical access. In order to reduce surface roughness, a subfloor made of pressed particle board was installed over the plywood. Two essential characteristics of the experimental facility are that it must provide a ZPG turbulent boundary with (1) very high $Re_{\tau}$ ($\equiv U_{\tau} \delta / \nu$) and (2) large boundary layer thickness thereby allowing measurements of high wall-normal spatial resolution. Although the experiments are performed in a low-speed tunnel, its large streamwise fetch allows high Reynolds numbers to be achieved ($Re_{x}$ as large as $9 \times 10^{6}$). This facility has been used to acquire ZPG turbulent boundary layer measurements at $Re_{\tau} \approx 3200$. Representative samples of these data are presented in the next section.

**TURBULENT BOUNDARY LAYER CHARACTERISTICS**

Figure 1a presents a sample turbulent boundary layer mean velocity profile obtained at the streamwise location of $x = 12.2$ m which corresponds to $Re_{\tau} \approx 3200$. The profile is presented using inner variable scaling with the Clauser method used to obtain the local friction velocity. The mean velocity profile is observed to exhibit a well-defined logarithmic region for $30 \leq y^{+} \leq 700$ in excellent agreement with classic log law of the wall. Figure 1b presents the corresponding streamwise component turbulence intensity profile using inner variable scaling. Consistent with previous studies, the profile exhibits a near-wall peak value of $u'^{2}/U_{\tau}^{2} = 7.25$ at $y^{+} \approx 15$. There is also indication of a weaker peak in the logarithmic region centered near $y^{+} \approx 250$.

Figure 2 presents the skewness factor profile, $S_{u} = u'^{3}/(u'^{2})^{3/2}$, as a function of $y^{+}$. This result is also in very good agreement with previous measurements in high-Reynolds number turbulent boundary layers. In particular, $S_{u}$ is positive in the near-wall region ($y^{+} \leq 20$) and becomes negative for $y^{+} \geq 130$. Note that the large-values of the skewness factor near the edge of the boundary layer are heavily influenced by intermittency associated with the turbulent / non-turbulent interface.

The spectral content of the ZPG turbulent boundary layer in streamwise component wavenumber domain was examined in terms of the pre-multiplied 1-D auto spectral density of the streamwise component fluctuation, $k_{x} \phi_{uu}/U_{\tau}^{2}$. This is presented in Figure 3a as a function of both $y^{+}$ and inner-variable scaled streamwise wave number, $\lambda_{x}^{+}$. The assumed convective speed to
convert to 1-D wavenumber domain was the local mean speed. The pre-multiplied energy spectra of the streamwise velocity fluctuation shows an inner peak located at $y^+ = 15$ and centered at $\lambda_x^+ = 1000$. A weaker outer peak appears near $y^+ \approx 250$, $y/\delta \approx 0.06$ with a much larger wavelength $\lambda_x^+ \approx 20000$, $\lambda_x/\delta \approx 5$. Figure 3b shows comparable ZPG turbulent boundary layer pre-multiplied energy wavenumber spectra as obtained by Mathis et al (2009). The inner and outer spectral peaks noted in Figure 3a are also apparent in this case. In fact, the outer peak is more apparent in Figure 3b and this is likely a consequence of the higher $Re_\tau$ of the Mathis et al (2009) experiment, $Re_\tau \approx 7300$.

**NATURE OF NEAR-WALL AMPLITUDE MODULATION**

As previously noted, small-scale velocity fluctuations in the near-wall region of the turbulent boundary layer undergo amplitude and phase modulation by larger-scales structures in the outer region. The correlation between near-wall amplitude modulation and large-scale structure in the boundary layer has been quantified in terms of an amplitude modulation correlation coefficient, $R$, which was defined by Mathis et al (2009) as,

$$ R = \frac{\bar{u_L^+} E_L(u_S^+)}{\sqrt{\bar{u_L^+}^2 E_L(u_S^+)^2}} $$

Here, $u_L^+$ and $u_S^+$ denote the inner-variable scaled large-scale and small-scale velocity fluctuations, respectively. $E_L(.)$ represents the envelope function of the near-wall fluctuations obtained using the Hilbert transform. In this experiment the velocity signal was divided into large and small-scale parts by using a cutoff frequency of 50 Hz which corresponds to one large eddy turnover timescale ($\delta/U_\infty$). Figure 4 presents the measured amplitude modulation correlation coefficient profile. It is found to be in remarkable agreement with R as measured by Mathis et al (2011) at similar Reynolds number. One can also note the striking similarity between the $R$ and $S_\delta$ profiles shown in Figs 4 and 2 respectively. In forming a scale-decomposed skewness, Mathis et al, 2011 showed that a constituent part of the skewness, $\bar{u_L^+} u_S^{+2}$ (where $\bar{X} = \bar{X}/(\bar{u}^2)^{1/2}$), has a profile shape similar to $R$ which is perhaps not surprising since the quantity $\bar{u_L^+} u_S^{+2}$ actually calculates the correlation between $u_L^+$ i.e. the large scales and $u_S^+$, a representative of the envelope of the small scales.

Since the amplitude modulation of the near wall fluctuations by structures in the outer region contributes significantly to skewness, it is also of interest to examine the spectral decomposition of the skewness of the near-wall signal. Similar to the spectral decomposition of $\bar{u}^2$ via the autospectral density one can also decompose the skewness in frequency domain using the higher order spectral estimate known as the bispectrum. The bispectrum is defined as,
modulation.
providing a new perspective on the nature of the near wall
regarding the modes undergoing nonlinear interaction thereby
of the skewness in frequency domain via (3) can reveal information
bispectrum obtained in the near-wall region at
related to the skewness by,
straightforward to show that the real part of the bispectrum is
result in difference modes at a frequency less than the interacting
frequency domain shown in Figure 5.

\[ E[u^3(t)] = \text{Re}(\sum_{f_1, f_2} B_{xxx}(f_1^*, f_2^*)) \]  

(3)
The real part of the bispectrum provides the contribution to the
skewness from spectral components \( f_1^* \) and \( f_2^* \). The decomposition
of the skewness in frequency domain via (3) can reveal information
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Spectral Decomposition of Skewness.

Due to symmetry properties, the bispectrum maps to \( f_1^* \) and \( f_2^* \) frequency domain shown in Figure 5.

\[ B_{xxx}(f_1^*, f_2^*) = E[X_f^* X_{f1} X_{f2}] \]  

(2)
where \( E[\cdot] \) denotes an expected value, \( X_f \) a Fourier transform and
the superscript \( C \) a complex conjugate. The frequencies have been
normalized by the large eddy frequency \( f_1^* = f_1 \delta/U_m \). It is
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domain as shown in Figure 5. The $y^+ = 15$ velocity fluctuation is considered as $x(t)$ and that at $y^+ = 200$ as $y(t)$. This figure shows phase coherence among wave triads only within a narrow band contained within the previously described region IIb. These interactions are of all of form $f_1^* - f_2^* = f_1^* - (f_1^* - \delta f^*) = \delta f^*$ where $\delta f^* \approx 0(0.5)$. This indicates that there is interaction between wave triads of two comparable and one much lower frequency. This might suggest that there is a phase locking between near wall small scale structure and larger structures of the outer layer. This result, along with the spectral decomposition of skewness presented earlier further underscores the importance of nonlinear mechanisms in characterizing the inner-outer layer interaction.

Figure 7. Cross bicoherence of the signal of the location $y^+ = 15$ and $y^+ \approx 200$.

A NONLINEAR SYSTEM MODEL

The polyspectral measurements presented in the previous section underscore the importance of nonlinear mechanisms in the interaction between the logarithmic and near wall regions of the turbulent boundary layer. In order to further characterize this interaction, experiments are performed using a measurement technique in which the nonlinear spetral dynamics characterizing this interaction is captured in terms of a transfer function containing both linear and quadratically nonlinear system elements. A similar approach has been used in plasma turbulence and in jet transition work (Thomas and Chu 1991). Two hot-wire probes are used for the measurement and their positioning is shown in Figure 9. The first probe is placed at $y^+ = 15$ and the inherent noise present in the measurement. By multiplying Eq. 6 by $X_f^*$ we can write an expression for linear transfer function

$$L_f = \frac{E[Y_f X_f^*] - \Sigma Q_{f_1^* f_2^*} E[X_{f_1^*} X_{f_2^*}]}{E[X_f X_f^*]}$$  \hspace{1cm} (7)$$

In a similar fashion the quadratic transfer functions can be computed by multiplying Eq. 6 with $X_f^* X_f^{C*}$ and taking an ensemble average

$$E \left[ Y_f X_f^{C*} X_f^{C*} \right] = L_f E[X_f X_f^{C*} X_f^{C*}] + \Sigma Q_{f_1^* f_2^*} E[X_{f_1^*} X_{f_2^*} X_{f_1^{C*}} X_{f_2^{C*}}] + E[\epsilon_f X_f^{C*} X_f^{C*}]$$  \hspace{1cm} (8)$$

where $f^* = f_1^* + f_2^*$ and $f_1^* = f_1' + f_2'$.

For a Gaussian process the equation (8) and (9) can be further simplified as third order moments are zero for such processes.

$$E[X_f X_f^{C*} X_f^{C*}] = 0$$  \hspace{1cm} (9)
For non-Gaussian processes the third order moments are non-zero and the examples have already been presented in the previous section (cross bispectrum). Equation (8) and (9) can be solved either by matrix method (Kim and Powers, 1988) or by iterative methods. In general the transfer functions can contain a large volume of information. Therefore it is useful to introduce normalized quantities to provide an easier interpretation of input-output relationship. A convenient tool is coherency. It measures the fraction of the power in output signal which can be accounted by local linear coherency, local quadratic coherency, mixed local coherency and error terms and is equal to unity.

\[ y^2(f^*) + y^2_r(f^*) + y^2_Q(f^*) + y^2_l(f^*) + \text{error} > = 1 \tag{10} \]

Local linear coherency measures fraction of power in the output signal at frequency \( f^* \) due to the linear transfer function.

\[ y^2_l(f^*) = |L_{f^*}|^2 E[X_f Y_{f^*}^*] \tag{11} \]

Local quadratic coherency measures fraction of power in the output signal at frequency \( f^* \) due to the quadratic transfer function.

\[ y^2_Q(f^*) = \sum_{f_1, f_2} |Q_{f_1, f_2}^2| \] \[ E[X_{f_1} Y_{f_2}^*] \tag{12} \]

Mixed local coherency measures fraction of output power that results from a non-zero auto bispectrum of input signal. It captures the nonlinear history of the flow at the system input.

\[ y^2_Q(f^*) = \frac{2 \text{Re}(L_{f^*} \sum_{f_1, f_2} [Q_{f_1, f_2}^2])}{E[X_{f_1} Y_{f_2}^*]} \tag{13} \]

And the error term

\[ y^2_l(f^*) = \frac{E[X_{f_1} Y_{f_2}^*]}{E[Y_{f_1} Y_{f_2}^*]} \tag{14} \]

Using these metrics the outer layer-inner layer interaction can be quantified for both reduced and enhanced drag conditions and can be compared to the natural flow.

**SUMMARY AND CONCLUSIONS**

Measurements are presented in a zero pressure gradient turbulent boundary layer at \( Re_t = 3200 \) which are in very good agreement with previous high Reynolds number experiments. Higher order spectra are used to characterize the nonlinear processes involved in the interaction between the near-wall and logarithmic regions. The frequency decomposition of the skewness of near wall fluctuations shows triad interactions between comparable frequencies and a much lower frequency that is characteristic of large-scale outer layer structure. In addition, two-point cross biocoherence measurements show nonlinear phase locking between modes in the near-wall and log regions. This provides motivation for a nonlinear system model of the inner-layer interaction. A methodology for obtaining the linear and quadratically nonlinear transfer functions and physical quantities derivable from them is described.

**REFERENCES**

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